Dual Schubert Polynomials

Eric Nie
Mentor: Pavel Galashin

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Inversions

Definition

For a permutation ω, we define $\ell(\omega) = |\{(\omega(i), \omega(j) | i < j \text{ and } \omega(i) > \omega(j)\}|$, which is also known as the **number of inversions**.

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Examples:

- $\ell(123) = 0$
- $\ell(132) = 1$
- $\ell(213) = 1$
- $\ell(231) = 2$
- $\ell(312) = 2$
- $\ell(321) = 3$

Lehmer Code

Definition

The **Lehmer code** of a permutation $\omega \in S_n$: $\text{code}(\omega) = (a_1, a_2, \ldots, a_n)$ where $a_i = |\{j | j > i \text{ and } \omega(i) > \omega(j)\}|$.

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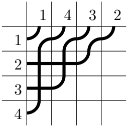
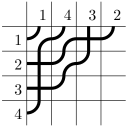
Definition

We denote the permutation in S_n with the most inversions by $\omega_0 = (n, n-1, \ldots, 1)$.

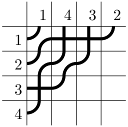
Pipe Dreams



$x_2^2x_3$



$x_1x_2x_3$



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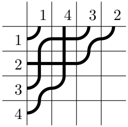
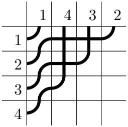


$x_1x_2^2$



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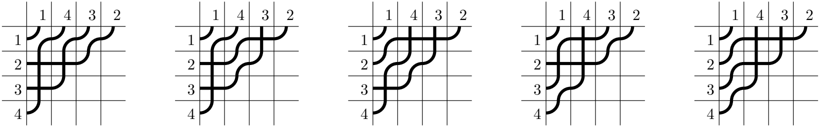
Possible Pipe Dreams of 1432

Pipe Dreams



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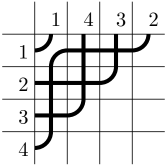

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Not a pipe dream

Schubert Polynomials

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with D taken over all the possible pipe dreams of ω.

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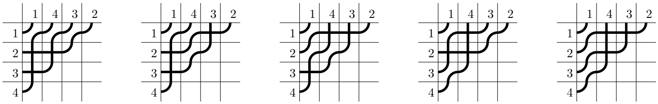

$x_2^2 x_3$ $\quad$ $x_1 x_2 x_3$ $\quad$ $x_1^2 x_3$ $\quad$ $x_1 x_2^2$ $\quad$ $x_1^2 x_2$

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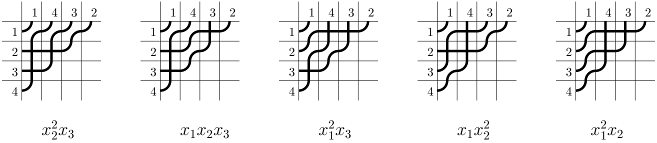

Possible Pipe Dreams of 1432

$$\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2.$$

Schubert-Kostka Matrix

Definition

We define the **Schubert-Kostka matrix** K as the coefficient matrix of the Schubert polynomial.

$$\mathfrak{S}_\omega = \sum_{a \in \mathbb{N}^n} K_{\omega,a} x^a.$$

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For $\omega = 1432$, we look at the entries of $K_{1432,a}$ for different a:

- $a \in \{(0,2,1,0),(1,1,1,0),(2,0,1,0),(1,2,0,0),(2,1,0,0)\}$

$$\implies K_{1432,a} = 1.$$

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$$\implies K_{1432,a} = 1.$$
- For any other a, we have $K_{1432,a} = 0$

Schubert-Kotska Matrix (continued)

Definition

Let K^{-1} be the inverse of the matrix K. It corresponds to the expansion of monomials in terms of Schubert polynomials

$$x^a = \sum_{\omega \in S_n} K^{-1}_{\omega,a} \mathfrak{S}_\omega.$$

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Definition

By taking the dual basis, we can define the dual Schubert polynomials in the following way:

$$\mathfrak{D}_\omega = \sum_{a \in \mathbb{N}^n} K^{-1}_{\omega,a} y^{(a)},$$

where the basis $\{y^{(a)} = \frac{y_1^{a_1}}{a_1!}\frac{y_2^{a_2}}{a_2!}\cdots\}$ is dual to the basis $\{x^a\}$.

Connection between Dual Schubert and Schubert polynomials (continued)

Lemma (Postnikov-Stanley, 2005)

For $\omega \in S_n$ and any $a \in \mathbb{N}^n$, we have

$$K^{-1}_{a,\omega} = \sum_{u \in S_n} (-1)^{\ell(u)} K_{\omega_0 \omega, u(\rho) - a}$$

where $\rho = (n-1, n-2, \ldots, 0) \in \mathbb{N}^n$.

Connection between Dual Schubert and Schubert polynomials (example)

$$K^{-1}_{a,\omega} = \sum_{u \in S_n} (-1)^{\ell(u)} K_{\omega_0\omega, u(\rho)-a}$$

For example, for $\omega = 4123$ we have $\omega_0\omega = 1432$.

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The only intersection of these two sets is 0210 when $u = id$. So,

$$K^{-1}_{3000,4123} = 1.$$

Pattern-avoiding Permutations

Definition

A permutation ω is σ-**avoiding** if there is no subsequence of ω with the same relative ordering as σ.

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An example of a non-213-avoiding permutation is $\omega = 23415$. The subsequence 215 has the same relative order as 213.

Proposition

For each $\sigma \in S_3$, there are C_n such permutations where C_n is the n-th Catalan number.

312-avoiding permutations

Proposition

For a 132-avoiding permutation, the Schubert polynomial is a single monomial $x^{\mathrm{code}(w)}$.

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Corollary (Postnikov-Stanley, 2005)

For a 312-avoiding permutation $w \in S_n$ *with* $c = code(\omega_0\omega)$, *and an arbitrary* $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, *we have*

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(u)} & \textit{if } a + c = u(\rho), \textit{ for some } u \in S_n, \\ 0 & \textit{otherwise.} \end{cases}$$

Symmetry of K^{-1}

The matrix K^{-1} has a certain symmetry that allows ω and $\omega_0\omega\omega_0$ to be interchanged. When we replace ω by $\omega_0\omega\omega_0$, the permutations are mapped from σ-avoiding to $\omega_0\sigma\omega_0$-avoiding permutations.

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Example

For $\sigma \in S_3$:

- $213 \leftrightarrow 132$
- $231 \leftrightarrow 312$
- $321 \leftrightarrow 321$
- $123 \leftrightarrow 123$

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Example

For $\sigma \in S_3$:

- $213 \leftrightarrow 132$
- $231 \leftrightarrow 312$
- $321 \leftrightarrow 321$
- $123 \leftrightarrow 123$

This gives a way to compute the entries of the matrix for a 231-avoiding permutation.

231-avoiding permutations

Corollary (Postnikov-Stanley, 2005)

For a 231-avoiding permutation $w \in S_n$ with $c = code(\omega\omega_0)$, and an arbitrary $a = (a_n, a_{n-1} \dots, a_1) \in \mathbb{N}^n$, we have

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(u)+|a|} & \text{if } a + c = u(\rho), \text{ for some } u \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

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Entries of $K_{a,w}^{-1}$

Recall that for ω which are 231- or 312-avoiding, the entries of $K_{a,\omega}^{-1}$ are equal to $-1, 0$, or $+1$.

Proposition

There exists an $\omega = 645231 \in S_6$ that is 123-avoiding, 213-avoiding, and 132-avoiding such that

$$K_{421042,\omega}^{-1} = -2.$$

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There is only one case left to check which is 321-avoiding permutations.

321-avoiding permutations

Conjecture

For a 321-avoiding permutation $\omega \in S_n$, $K^{-1}_{a,\omega}$ is equal to -1,0 or 1 for all $a \in \mathbb{N}^n$.

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For a 321-avoiding permutation $\omega \in S_n$, $K^{-1}_{a,\omega}$ *is equal to -1,0 or 1 for all* $a \in \mathbb{N}^n$.

Lemma

For a 321-avoiding permutation $\omega \in S_n$ *we have that* $K^{-1}_{a,\omega} = 0$ *for* $a >_{lex} \mathrm{code}(\omega)$ *where* $>_{lex}$ *is the lexicographic order.*

321-avoiding permutations

Conjecture

For a 321-avoiding permutation $\omega \in S_n$, $K^{-1}_{a,\omega}$ is equal to -1,0 or 1 for all $a \in \mathbb{N}^n$.

Lemma

For a 321-avoiding permutation $\omega \in S_n$ we have that $K^{-1}_{a,\omega} = 0$ for $a >_{lex} \mathrm{code}(\omega)$ where $>_{lex}$ is the lexicographic order.

Example

$$K^{-1}_{1200,2341} = 0$$

Since, $1200 >_{lex} \mathrm{code}(\omega) = 1110$.

Dominance Order

Definition

For $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, $a \geq_D b$ if and only if $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k$ for all $k \geq 1$.

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Lemma

The coefficient can be nonzero only if $code(\omega) \geq_D a \geq_D code(\omega_0 \omega \omega_0)$.

Example

For $\omega = 2341$, we have $\mathrm{code}(\omega) = 1110$.
For $a = 0300$, $a <_{lex} \mathrm{code}(\omega)$, but we do not have $a <_D \mathrm{code}(\omega)$.
So,

$$K^{-1}_{a,\omega} = 0.$$

Future Research

- Find a formula for Grassmannian permutations which form a subset of 321-avoiding permutations.

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- Find a formula for Grassmannian permutations which form a subset of 321-avoiding permutations.
- Prove the conjecture above: for $\omega \in S_n$, $K^{-1}_{a,\omega}$ is equal to -1, 0, or 1 for all 321-avoiding permutations

Acknowledgements

I would like to acknowledge the following people:

- Prof. Alexander Postnikov
- My mentor Pavel Galashin
- The MIT PRIMES Program
- My parents